Single Mode Coherent Synchrotron Radiation Instability [a]

S. Heifets, G. Stupakov

Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309, USA *October 25, 2002*

1 Abstract

The coherent synchrotron radiation (CSR) instability is studied in the case where it is caused by a single synchronous component of the field excited by the beam in a toroidal wave-guide. Parameters of a storage ring can be chosen in a such a way that, due to the shielding effect, one CSR mode determines beam instability. Such a regime is different from the regime studied before (SH,GS) where the continuous CSR spectrum was implied. The beam dynamics for the single-mode-instability regime has common features with the FEL theory and may be advantageous for a machine designed as a CSR source of radiation.

[a] Supported by Department of Energy contract DOE–AC03–76SF00515.

2 Introduction

• CSR exists only for $k = \omega/c > k_0 = \frac{\pi}{a}\sqrt{R/a}$ where the long bunch spectrum is suppressed.

• The instability takes place within a range of the wave lengths which depends on beam current and machine parameters.

• Parameters may be chosen in such a way that only few CSR modes close to the shielding threshold are important.

• For such k, the wave guide modes are discrete, the CSR impedance is a sum of δ-functions, and the previous analysis may be invalid.

• We analyze the CSR instability driven by a single wave-guide mode. In this context, it is a very general problem arising in plasma, FEL theory, and microwave instability.

• Plan:

a) Summary on the toroidal wave-guide modes

b) Beam-EM interaction from scratch

c) Linear theory of the instability

d) Nonlinear stage of the instability

• Geometry: toroidal chamber with square cross-section and constant curvature R.

Here some results for the synchronous modes (Stupakov+Kotelnikov):

• For the lowest synchronous mode

$$\frac{\omega_1}{c} = \frac{4.78}{\pi} k_0, \quad k_0 = \frac{\pi}{a}\sqrt{\frac{R}{a}}$$
$$\chi = \frac{4.94}{a^2}, \quad \frac{v_g}{c} = 1 - 0.62\,\frac{a}{R}.$$

• For the next mode:

$(\omega/c) = (8.78/\pi)k_0$, $\chi = 3.01/a^2$.

• The higher order modes are numbered by the integers p, m:

$$q_n = \frac{\omega_n}{c} = k_0\sqrt{p^2 + m^{2/3}(\frac{3k^2}{k_0^2})^{2/3}}$$
$$\chi_n(p,m) \propto (\frac{2\pi}{a})^2(\frac{k_0}{k})^{4/3}\frac{p^2}{m^{1/3}}.$$

• For $p > p_{cr}$ modes are suppressed, $p_{cr} \simeq (\frac{k}{k_0})^{2/3}$.

3 Beam stability

• The ω-component of the field excited by the beam is given by the superposition of the eigen modes,

$$\vec{E}_\omega(\vec{r},s) = \sum_n \vec{e}_n(\vec{r}) \int \frac{dq}{2\pi} e^{iqs} C_n(q,\omega).$$

• For modulation $k\sigma_l >> 1$, bunch is treated as coasting beam with $n_b = N_b/l_b$ neglecting transverse emittance.

• The distribution function and beam current are defined by the amplitudes g,

$$f(z,\delta,t) = \int \frac{d\omega dq}{(2\pi)^2} e^{i(qz-\omega t)} g(\omega,q,\delta)$$

$$j_\omega(s) = ecN_{tot} \int \frac{dq}{2\pi} e^{iqs} \int d\delta g(\omega - qc, q, \delta).$$

• Amplitudes C_n and g are related by

The Lorentz identity (Jackson, Vainstein)

$$\int dS(E_\omega \times H_n - E_n \times H_\omega) = Z_0 \int dV \vec{j}_\omega.\vec{E}_n,$$

and linearized Vlasov equation.

• That gives the set of coupled equations:

$$C_n(q,\omega) = i\frac{Z_0}{N_n}(\vec{e_n^*}(0).\vec{s})\frac{ecN_{tot}}{q-q(n,\omega)-i\epsilon}\int d\delta g(\omega{-}qc,q,\delta),$$

$$(\omega{+}\eta c\delta q)g(\omega,q,\delta) = -i\frac{\partial f_0}{\partial \delta}(\frac{ec}{LE_0})\sum_n(\vec{s}.\vec{e}_n(0))C_n(q,\omega{+}qc),$$

where the norm N_n is proportional to the power in the mode.

In particular, the energy variation

$$\frac{d\delta(z,t)}{dt} = \frac{e\,c}{E_0}\vec{s}.\vec{E}(0, s = ct + z, t),$$

or

$$\frac{d\delta(z,t)}{dc\,t} = -Z_0 \frac{r_e N_{tot}, v_g}{\gamma(1-\beta_g)} \frac{|\vec{e}_n(0).\vec{s}|^2}{N_n}$$
$$\int dz' d\delta' f[z', \delta', t - \frac{z'-z}{c(1-\beta_g)}] e^{-iq_n(z'-z)}.$$

Note:the delay time f depends on the group velocity.

That relates the loss factor and the amplitudes $\vec{e}$,

$$\chi_n = Z_0 \frac{v_{g,n}}{(1-\beta_{g,n})} \frac{|(\vec{s}.\vec{e}_n(0))|^2}{N_n}.$$

Exclude one of the amplitudes, we get

$g(\omega, q, \delta) \propto= G(\omega, q)(\partial f_0/\partial\delta)$.

The EM amplitude $G(\omega, q) \neq 0$ only for ω defined by the d.eq.

$$1 = \sum_n \frac{\lambda_n \delta_0}{q - q(n, \omega + qc) + i\epsilon} \int d\delta \frac{\frac{\partial f_0}{\partial \delta}}{\omega + \eta c q \delta},$$

where $\int d\delta f_0(\delta) = 1$, and

$$\lambda_n = \frac{r_e c^2 n_b}{\gamma \delta_0 v_{g,n}} (1 - \beta_{g,n}) \chi_n.$$

Note that

$$f(z, \delta, t) \propto e^{i(qz - \omega t)},$$

and EM wave

$$E \propto G\, e^{i(qs - \bar{\omega} t)}$$

propagating in the beam pipe with the frequency

$\bar{\omega} = \omega + qc$.

4 Dispersion relation for a single mode

For a single EM mode with frequency $\bar{\omega} = \omega + qc$ in the laboratory frame, the dispersion equation takes the form

$$q - q(n,\bar{\omega}) = \lambda_n \int \frac{dp\,(d\rho_0(p)/dp)}{\bar{\omega} - qc + (\eta c \delta_0 q)p + i\epsilon}.$$

Let us assume that

$$\bar{\omega} = \omega_n + \Omega, \quad \Omega << \omega_n.$$

Then, $q(n,\bar{\omega}) = q_n + \frac{\Omega}{v_{g,n}}$,

If $\Omega >> (\eta \omega_n \delta_0)$, the dispersion equation is reduced to the cubic equation:

$$[\Omega - (q - q_n)v_{g,n}][\Omega - (q - q_n)c]^2 = -\lambda_n v_{g,n} \eta \delta_0 \omega_n.$$

At the maximum,

$$\Omega_{q=q_n} = \mu\, e^{i\pi/3},$$
$$\mu = c[\frac{r_e n_b}{\gamma}(1 - \beta_g)\chi \frac{\omega_n}{c}]^{1/3}.$$

Numeric solution of the d.eq. is plotted in Fig. 1 in dimensionless form using

$$x = \frac{\Omega - (q - q_n)c}{\mu}, \quad \Delta q = \frac{c(q - q_n)(1 - \beta_{g,n})}{\mu}.$$

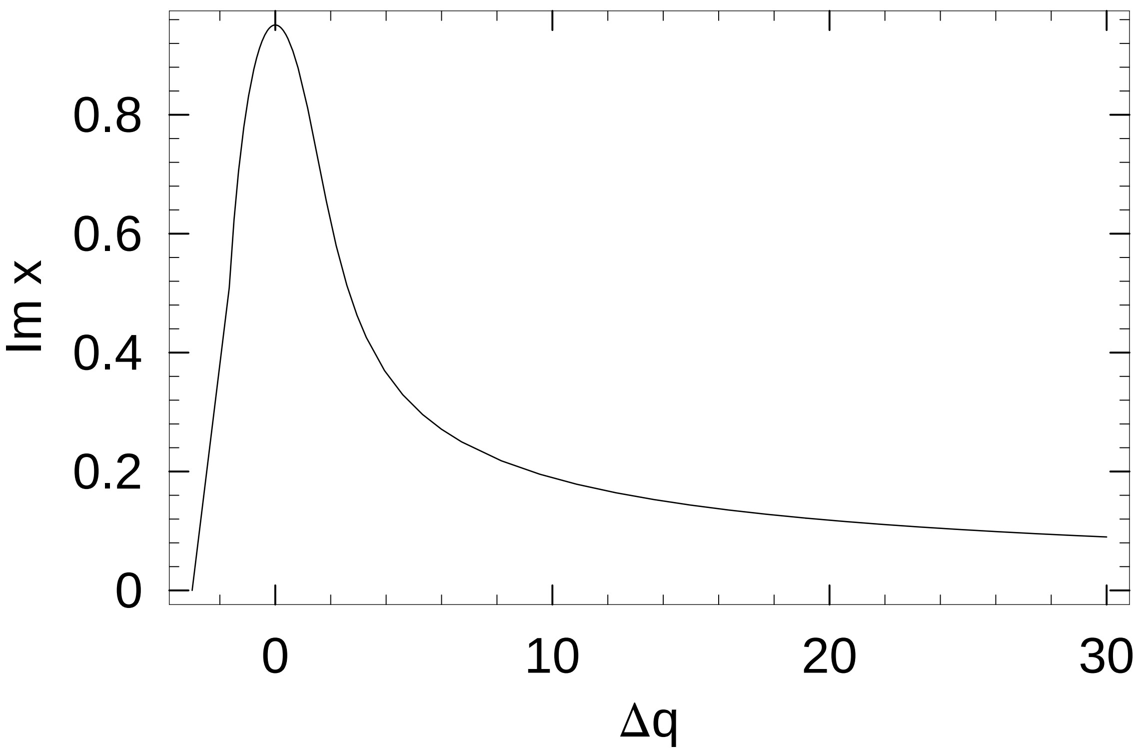

Figure 1: Dimensionless growth rate Im[x] vs. Δq, see text.

Table 1: **Growth rate in four machines**

Parameter	units	LER,PEP-II	ALS	VUV NSLS	HER,PEP-II
linear bunch density	n_b, 10^{10} 1/cm	2.65	7.00	3.60	8.26
energy	GeV	3.1	1.5	8.1	9.0
η	10^{-3}	1.31	1.41	2.35	2.1
δ_0	10^{-4}	8.1	7.1	5.0	6.1
vertical gap (square pipe)	b, cm	4.5	2	4.2	4.5
curvature	ρ, m	13.7	4.0	19.1	165.0
mode frequency	f, 10^{10}, Hz	8.85	16.14	3.66	30.71
mode loss factor	χ, V/pC	0.22	1.11	0.25	0.22
$1 - v_g/c$	10^{-3}	1.03	3.1	13.6	0.17
μ	$1/\mu s$	8.1	31.9	22.6	2.7
n_{cr}	10^{10} 1/cm	45.0	14.3	2.46	490.1
n_{bunch}	10^{10} 1/cm	3.65	7.0	3.6	0.83

5 Transition to the continuous spectrum.

• Interaction with individual CSR makes sense until Ω is small compared to the distance between the modes $\Delta\omega$.

• The coherent frequency shift increases with $k = \omega_n/c$,

$$\frac{\Omega}{c} \simeq [\frac{\eta r_e n_b k}{\gamma}(1-\beta_{g,n})\chi]^{1/3}.$$

• If modes overlap, the loss factor has to be replaced by the averaged loss factor weighted with the density of resonances $\langle \chi_n dN/dk \rangle$,

$$\langle \chi_n(p,m)\frac{dN}{dk}\rangle = \frac{(2\pi}{a})^2(\frac{k_0}{k})^{4/3}$$
$$\int dp\, dm \frac{p^2}{m^{1/3}}\delta[k - k_0\sqrt{p^2 + m^{2/3}(\frac{3k^2}{k_0^2})^{2/3}}],$$

Calculations give result $\propto dP/d\omega \propto \mathrm{Re}Z_{CSR}(\omega)$,

$$Re[Z(\omega)] = 1.6\ Z_0(kR)^{1/3}.$$

• The overlapping takes place at

$$n_{cr} \equiv \frac{\pi\gamma\delta_0}{r_e}\,(\eta\delta_0)^{3/5}\,(\frac{R}{a})^{3/5}.$$

• The coherent frequency for a cold beam obtained before in the case of the continuous spectrum matches the single mode result at the overlapping.

6 Nonlinear regime

• The growth rate of the instability is large and hardly can be observed experimentally.

• The state of the beam is defined by the non-linear regime of the instability described by the Fokker-Plank equation.

• Let us change variables

$$\{t, z, \delta\}, V \rightarrow \{\tau, \zeta, P\}, A(\tau),$$

$$t = -\frac{\tau}{\mu}, \quad z = \frac{\zeta}{q_n}, \quad \delta = \frac{\mu}{\eta\omega_n} P, \quad V = \frac{\mu^2}{\eta\omega_n} A(\tau) e^{iq_n z}.$$

The F-Pl. equation then is

$$\frac{\partial F}{\partial \tau} + P\frac{\partial F}{\partial \zeta} + [A(\tau)e^{i\zeta} + c.c.]\frac{\partial F}{\partial P} = \Gamma \frac{\partial}{\partial P}[\Delta^2 \frac{\partial F}{\partial P} + PF],$$

where

$$\Gamma = \frac{\gamma_{SR}}{\mu}, \quad \Delta = \frac{\eta\omega_n\delta_0}{\mu}, \quad \int d\zeta dPF(\zeta, P, \tau) = 1.$$

- $A(\tau)$ is defined by

$$\frac{dA(\tau)}{d\tau} = \langle e^{-i\zeta} \rangle + iA_0 A$$

where

$$\langle e^{-i\zeta} \rangle = \int d\zeta dP \, F(\zeta, P, \tau) \, e^{-i\zeta}, \quad A_0 = \frac{\omega_n (1 - \beta_g)}{\mu}.$$

- Results of the linear approximation take the form

$$F(\zeta, P, \tau) = \frac{F_0(P)}{L} + [F_1(P) e^{i(\zeta - \nu\tau)} + c.c.].$$

The steady-state distribution without the perturbation is

$$F_0(P) = \frac{1}{\sqrt{2\pi}} e^{-\frac{P^2}{2\Delta^2}},$$

If $\nu >> \Delta$, the dispersion equation is simplified to

$$\nu^2 (\nu + A_0) = -1,$$

The growth rate $\mathrm{Im}\nu \simeq 1$ for $\nu >> A_0$ and $\mathrm{Im}\nu = 1/\sqrt{A_0}$ otherwise.

6.1 Quasi-linear regime

• The exponential growth of the linear theory saturates due to the effect of the growing mode on the distribution of particles.

• Assuming that $F_1 \propto G e^{-i\nu\tau}$ varies much faster than F_0, we preserve the dispersion equation allowing adiabatic variation of F_0 with time.

• Using explicit form of $F_1(P)$ and the dispersion equation, equation for F_0 is transformed to the diffusion-like equation (Chin-Yokoya, 1984):

$$\frac{\partial F_0}{\partial \tau} = \frac{\partial}{\partial P}[\Delta_{eff}^2 \frac{\partial F_0}{\partial P} + P F_0],$$

where

$$\Delta_{eff}^2 = \Gamma \Delta^2 + \frac{2\nu_2 |G|^2}{(\nu_1 - P)^2 + \nu_2^2} e^{2\nu_2 \tau},$$

where $\nu = \nu_1 + i\nu_2$.

The energy spread increases in time as $< P^2 >= 2\Delta_{eff}^2 \tau$. When $< P^2 > \simeq \nu^2 \simeq 1$, Landau damping slows down the growth rate. That happens at

$$\tau \simeq \ln \frac{1}{|G|^2}, \quad t \simeq 1/\mu.$$

6.2 Analogy with FEL. Simulations

• The quasi-linear theory clarifies the mechanism of transition to saturation but does not tell us what happens later, at $\tau >> 1$.

• To study this stage, we replace the F.-Pl. equation by the set of equations for particles and the coherent mode

$$\frac{d\zeta}{d\tau} = P, \quad \frac{dP}{d\tau} = [A(\tau)e^{i\zeta} + c.c.] - \Gamma P + \kappa(\tau),$$
$$\frac{dA(\tau)}{d\tau} = \langle e^{-i\zeta} \rangle + iA_0 A$$

Here damping and random force $\kappa(\tau)$ are included,

$$\langle \kappa \rangle = 0, \quad \langle \kappa(\tau)\kappa(\tau') \rangle = 2\Gamma\Delta^2\,\delta(\tau - \tau').$$

The system has a universal form of beam-wave interaction and exactly equivalent to similar problem of the FEL theory but with additional damping and random force.

Parameter μ is equivalent to the Pierce parameter ρ in the FEL theory.

The system of equations is very suitable for numeric simulations. We reproduced results of simulations (Gluckstern, Krinsky, Okamoto, 1993) at relatively small time using MATHEMATICA, see Fig. 3.

The amplitude of perturbation oscillates after initial exponential growth.

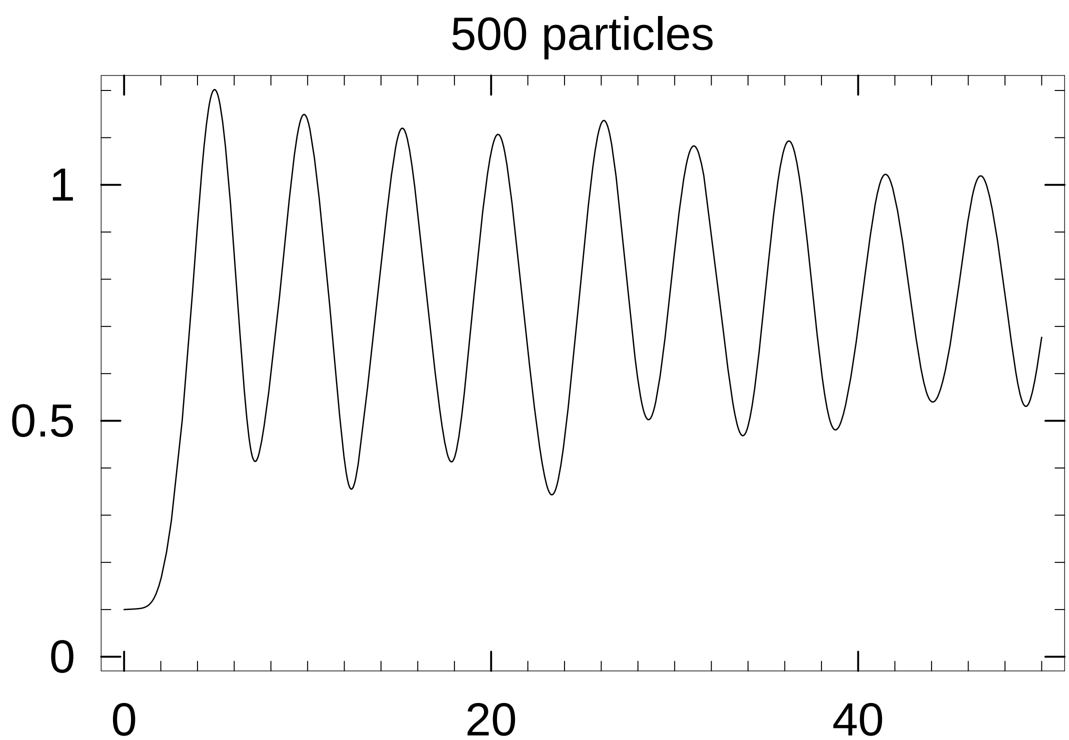

Figure 2: Amplitude dependence on time in the nonlinear regime of the instability. After $\tau \simeq 1$, exponential growth of the linear regime changes changes to oscillations with the average amplitude $A \simeq 1$ and frequency $\nu \simeq \sqrt{2|A|} \simeq 1$.

6.3 Asymptotic behavior. Resonance solution

• Unfortunately, contrary to the FEL theory, for a beam in the storage ring, $t >> 1/\gamma_{SR}$, and result shown in Fig. 3 are not sufficient for this purpose.

• For the FEL, the asymptotic behavior was analyzed by (G-K-O, 1993).

Analysis does not include damping and quantum fluctuations and substantially based on the integrals of motion.

The later are not conserved if are included, for example

$$\frac{d}{d\tau}[|A|^2 + \langle P \rangle] = -\Gamma \langle P \rangle.$$

• Simulations including damping and fluctuations are in progress.

We also study some analytic approach to the long-term behavior of the system.

• It should be noted, however, that the growing spread of frequencies in the beam may generate new resonances.

Instead of the regular behavior in the single mode regime (low beam densities) the separatrices of interacting resonances may interact and time dependence of the mode amplitudes may become irregular including bursts of the CSR.

• Analysis of transition of the single-mode regime to the regime of continuous spectrum of the CSR modes gives, therefore, the estimate for the maximum beam density where the dynamics is defined by a single CSR mode.

• Parameter μ gives the estimate for the time to reach saturation. In the case of the interacting modes, the same parameter may give the growth time of the burst which is followed by the relaxation to the initial state defined by the smallest of $1/\gamma_{SR}$ or inverse frequency spread.

7 Conclusion

• The CSR instability close to the shielding threshold may be defined by the interaction with a single synchronous CSR mode.

• Although the impedance in the case of the perfectly conducting wall is singular, the dispersion equation does not have any singularity.

• It is shown that in the linear regime for typical machine parameters, the instability has large growth rate.

• The growth rate is independent of the energy spread in the beam but is limited by the Landau damping.

• Analysis of the beam interacting with a single mode is quite analogous with similar systems such as FEL.

• The analysis of the nonlinear stage is presented which may have the universal character.

Next Steps to Study:

1. Tracking simulations at large time.
2. Interaction of 2 Modes
3. Non-ideal ring